90 Minutes—No Calculator

Notes: (1) In this examination, ln x denotes the natural logarithm of x (that is, logarithm to the base e).

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. The area of the region in the first quadrant enclosed by the graph of $y = x(1-x)$ and the x-axis is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{5}{6}$ (E) 1

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2. $\int_0^1 x\left(x^2+2\right)^2 dx =$

(A) $\frac{19}{2}$ (B) $\frac{19}{3}$ (C) $\frac{9}{2}$ (D) $\frac{19}{6}$ (E) $\frac{1}{6}$

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3. If $f(x) = \ln\left(\sqrt{x}\right)$, then $f''(x) =$

(A) $-\frac{2}{x^2}$ (B) $-\frac{1}{2x^2}$ (C) $-\frac{1}{2x}$ (D) $-\frac{1}{2x^{\frac{3}{2}}}$ (E) $\frac{2}{x^2}$

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4. If u, v, and w are nonzero differentiable functions, then the derivative of $\frac{uv}{w}$ is

(A) $\frac{uv'+u'v}{w'}$ (B) $\frac{u'v'w-uvw'}{w^2}$ (C) $\frac{uvw'-uv'w-u'vw}{w^2}$

(D) $\frac{u'vw+uv'w+uvw'}{w^2}$ (E) $\frac{uv'w+u'vw-uvw'}{w^2}$

5. Let f be the function defined by the following.

$$f(x)=\begin{cases} \sin x, & x<0 \\ x^2, & 0\le x<1 \\ 2-x, & 1\le x<2 \\ x-3, & x\ge 2 \end{cases}$$

For what values of x is f NOT continuous?

(A) 0 only (B) 1 only (C) 2 only (D) 0 and 2 only (E) 0, 1, and 2

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6. If $y^2-2xy=16$, then $\frac{dy}{dx}=$

(A) $\frac{x}{y-x}$ (B) $\frac{y}{x-y}$ (C) $\frac{y}{y-x}$ (D) $\frac{y}{2y-x}$ (E) $\frac{2y}{x-y}$

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7. $\int_2^{+\infty}\frac{dx}{x^2}$ is

(A) $\frac{1}{2}$ (B) $\ln 2$ (C) 1 (D) 2 (E) nonexistent

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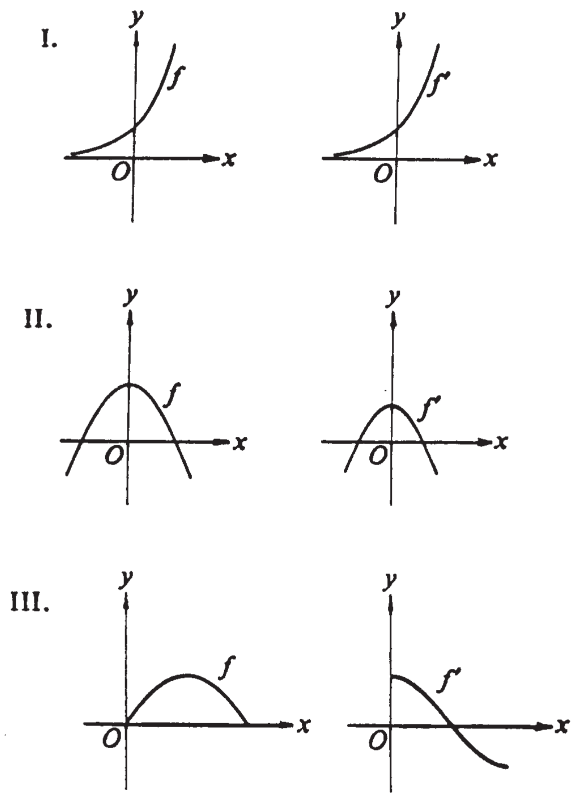
8. If $f(x)=e^x$, then $\ln\left(f'(2)\right)=$

(A) 2 (B) 0 (C) $\frac{1}{e^2}$ (D) $2e$ (E) e^2

9. Which of the following pairs of graphs could represent the graph of a function and the graph of its derivative?



(A) I only (B) II only (C) III only (D) I and III (E) II and III

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10. $\lim\limits_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h}$ is

(A) 0 (B) 1 (C) $\sin x$ (D) $\cos x$ (E) nonexistent

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11. If $x + 7y = 29$ is an equation of the line <u>normal</u> to the graph of f at the point $(1, 4)$, then $f'(1) =$

(A) 7 (B) $\frac{1}{7}$ (C) $-\frac{1}{7}$ (D) $-\frac{7}{29}$ (E) -7

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12. A particle travels in a straight line with a constant acceleration of 3 meters per second per second. If the velocity of the particle is 10 meters per second at time 2 seconds, how far does the particle travel during the time interval when its velocity increases from 4 meters per second to 10 meters per second?

(A) 20 m (B) 14 m (C) 7 m (D) 6 m (E) 3 m

13. $\sin(2x) =$

(A) $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots + \dfrac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + \ldots$

(B) $2x - \dfrac{(2x)^3}{3!} + \dfrac{(2x)^5}{5!} - \ldots + \dfrac{(-1)^{n-1} (2x)^{2n-1}}{(2n-1)!} + \ldots$

(C) $-\dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} - \ldots + \dfrac{(-1)^{n} (2x)^{2n}}{(2n)!} + \ldots$

(D) $\dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \ldots + \dfrac{x^{2n}}{(2n)!} + \ldots$

(E) $2x + \dfrac{(2x)^3}{3!} + \dfrac{(2x)^5}{5!} + \ldots + \dfrac{(2x)^{2n-1}}{(2n-1)!} + \ldots$

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14. If $F(x) = \int_1^{x^2} \sqrt{1+t^3}\, dt$, then $F'(x) =$

(A) $2x\sqrt{1+x^6}$

(B) $2x\sqrt{1+x^3}$

(C) $\sqrt{1+x^6}$

(D) $\sqrt{1+x^3}$

(E) $\displaystyle\int_1^{x^2} \frac{3t^2}{2\sqrt{1+t^3}}\, dt$

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15. For any time $t \geq 0$, if the position of a particle in the *xy*-plane is given by $x = t^2 + 1$ and $y = \ln(2t+3)$, then the acceleration vector is

(A) $\left(2t, \dfrac{2}{(2t+3)}\right)$

(B) $\left(2t, \dfrac{-4}{(2t+3)^2}\right)$

(C) $\left(2, \dfrac{4}{(2t+3)^2}\right)$

(D) $\left(2, \dfrac{2}{(2t+3)^2}\right)$

(E) $\left(2, \dfrac{-4}{(2t+3)^2}\right)$

16. $\int xe^{2x}dx =$

(A) $\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$ (B) $\frac{xe^{2x}}{2} - \frac{e^{2x}}{2} + C$ (C) $\frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + C$

(D) $\frac{xe^{2x}}{2} + \frac{e^{2x}}{2} + C$ (E) $\frac{x^2e^{2x}}{4} + C$

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17. $\int_2^3 \frac{3}{(x-1)(x+2)}dx =$

(A) $-\frac{33}{20}$ (B) $-\frac{9}{20}$ (C) $\ln\left(\frac{5}{2}\right)$ (D) $\ln\left(\frac{8}{5}\right)$ (E) $\ln\left(\frac{2}{5}\right)$

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18. If three equal subdivisions of $[-4, 2]$ are used, what is the trapezoidal approximation of $\int_{-4}^{2} \frac{e^{-x}}{2}dx$?

(A) $e^2 + e^0 + e^{-2}$ (B) $e^4 + e^2 + e^0$ (C) $e^4 + 2e^2 + 2e^0 + e^{-2}$

(D) $\frac{1}{2}\left(e^4 + e^2 + e^0 + e^{-2}\right)$ (E) $\frac{1}{2}\left(e^4 + 2e^2 + 2e^0 + e^{-2}\right)$

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19. A polynomial $p(x)$ has a relative maximum at $(-2,4)$, a relative minimum at $(1,1)$, a relative maximum at $(5,7)$ and no other critical points. How many zeros does $p(x)$ have?

(A) One (B) Two (C) Three (D) Four (E) Five

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20. The statement "$\lim_{x\to a} f(x) = L$" means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

(A) if $0 < |x-a| < \varepsilon$, then $|f(x)-L| < \delta$

(B) if $0 < |f(x)-L| < \varepsilon$, then $|x-a| < \delta$

(C) if $|f(x)-L| < \delta$, then $0 < |x-a| < \varepsilon$

(D) $0 < |x-a| < \delta$ and $|f(x)-L| < \varepsilon$

(E) if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$

21. The average value of $\dfrac{1}{x}$ on the closed interval $[1,3]$ is

(A) $\dfrac{1}{2}$ (B) $\dfrac{2}{3}$ (C) $\dfrac{\ln 2}{2}$ (D) $\dfrac{\ln 3}{2}$ (E) $\ln 3$

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22. If $f(x) = \left(x^2+1\right)^x$, then $f'(x) =$

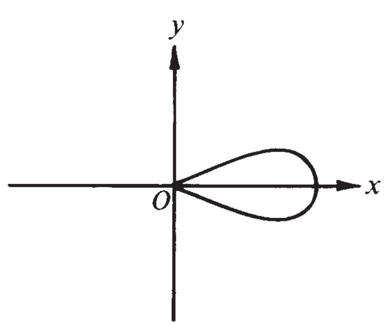
(A) $x\left(x^2+1\right)^{x-1}$

(B) $2x^2\left(x^2+1\right)^{x-1}$

(C) $x\ln\left(x^2+1\right)$

(D) $\ln\left(x^2+1\right) + \dfrac{2x^2}{x^2+1}$

(E) $\left(x^2+1\right)^x\left[\ln\left(x^2+1\right) + \dfrac{2x^2}{x^2+1}\right]$



23. Which of the following gives the area of the region enclosed by the loop of the graph of the polar curve $r = 4\cos(3\theta)$ shown in the figure above?

(A) $16\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos(3\theta)\,d\theta$ (B) $8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos(3\theta)\,d\theta$ (C) $8\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2(3\theta)\,d\theta$

(D) $16\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\,d\theta$ (E) $8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\,d\theta$

24. If c is the number that satisfies the conclusion of the Mean Value Theorem for $f(x) = x^3 - 2x^2$ on the interval $0 \le x \le 2$, then $c =$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{4}{3}$ (E) 2

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25. The base of a solid is the region in the first quadrant enclosed by the parabola $y = 4x^2$, the line $x = 1$, and the x-axis. Each plane section of the solid perpendicular to the x-axis is a square. The volume of the solid is

(A) $\frac{4\pi}{3}$ (B) $\frac{16\pi}{5}$ (C) $\frac{4}{3}$ (D) $\frac{16}{5}$ (E) $\frac{64}{5}$

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26. If f is a function such that $f'(x)$ exists for all x and $f(x) > 0$ for all x, which of the following is NOT necessarily true?

(A) $\int_{-1}^{1} f(x)\,dx > 0$

(B) $\int_{-1}^{1} 2f(x)\,dx = 2\int_{-1}^{1} f(x)\,dx$

(C) $\int_{-1}^{1} f(x)\,dx = 2\int_{0}^{1} f(x)\,dx$

(D) $\int_{-1}^{1} f(x)\,dx = -\int_{1}^{-1} f(x)\,dx$

(E) $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

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27. If the graph of $y = x^3 + ax^2 + bx - 4$ has a point of inflection at $(1, -6)$, what is the value of b?

(A) –3 (B) 0 (C) 1 (D) 3

(E) It cannot be determined from the information given.

28. $\dfrac{d}{dx}\ln\left|\cos\left(\dfrac{\pi}{x}\right)\right|$ is

(A) $\dfrac{-\pi}{x^2\cos\left(\dfrac{\pi}{x}\right)}$ (B) $-\tan\left(\dfrac{\pi}{x}\right)$ (C) $\dfrac{1}{\cos\left(\dfrac{\pi}{x}\right)}$

(D) $\dfrac{\pi}{x}\tan\left(\dfrac{\pi}{x}\right)$ (E) $\dfrac{\pi}{x^2}\tan\left(\dfrac{\pi}{x}\right)$

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29. The region R in the first quadrant is enclosed by the lines $x = 0$ and $y = 5$ and the graph of $y = x^2 + 1$. The volume of the solid generated when R is revolved about the y-axis is

(A) 6π (B) 8π (C) $\dfrac{34\pi}{3}$ (D) 16π (E) $\dfrac{544\pi}{15}$

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30. $\displaystyle\sum_{i=n}^{\infty}\left(\frac{1}{3}\right)^i =$

(A) $\dfrac{3}{2}-\left(\dfrac{1}{3}\right)^n$ (B) $\dfrac{3}{2}\left[1-\left(\dfrac{1}{3}\right)^n\right]$ (C) $\dfrac{3}{2}\left(\dfrac{1}{3}\right)^n$

(D) $\dfrac{2}{3}\left(\dfrac{1}{3}\right)^n$ (E) $\dfrac{2}{3}\left(\dfrac{1}{3}\right)^{n+1}$

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31. $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx =$

(A) $\dfrac{8}{3}$ (B) $\dfrac{16}{3}$ (C) π (D) 2π (E) 4π

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32. The general solution of the differential equation $y' = y + x^2$ is $y =$

(A) Ce^x (B) $Ce^x + x^2$ (C) $-x^2 - 2x - 2 + C$

(D) $e^x - x^2 - 2x - 2 + C$ (E) $Ce^x - x^2 - 2x - 2$

33. The length of the curve $y = x^3$ from $x = 0$ to $x = 2$ is given by

(A) $\int_0^2 \sqrt{1+x^6}\,dx$ (B) $\int_0^2 \sqrt{1+3x^2}\,dx$ (C) $\pi\int_0^2 \sqrt{1+9x^4}\,dx$

(D) $2\pi\int_0^2 \sqrt{1+9x^4}\,dx$ (E) $\int_0^2 \sqrt{1+9x^4}\,dx$

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34. A curve in the plane is defined parametrically by the equations $x = t^3 + t$ and $y = t^4 + 2t^2$. An equation of the line tangent to the curve at $t = 1$ is

(A) $y = 2x$ (B) $y = 8x$ (C) $y = 2x - 1$

(D) $y = 4x - 5$ (E) $y = 8x + 13$

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35. If k is a positive integer, then $\lim_{x\to+\infty} \frac{x^k}{e^x}$ is

(A) 0 (B) 1 (C) e (D) $k!$ (E) nonexistent

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36. Let R be the region between the graphs of $y = 1$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$. The volume of the solid obtained by revolving R about the x-axis is given by

(A) $2\pi\int_0^{\frac{\pi}{2}} x\sin x\,dx$ (B) $2\pi\int_0^{\frac{\pi}{2}} x\cos x\,dx$ (C) $\pi\int_0^{\frac{\pi}{2}} (1-\sin x)^2\,dx$

(D) $\pi\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ (E) $\pi\int_0^{\frac{\pi}{2}} \left(1-\sin^2 x\right)dx$

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37. A person 2 meters tall walks directly away from a streetlight that is 8 meters above the ground. If the person is walking at a constant rate and the person's shadow is lengthening at the rate of $\frac{4}{9}$ meter per second, at what rate, in meters per second, is the person walking?

(A) $\frac{4}{27}$ (B) $\frac{4}{9}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$ (E) $\frac{16}{9}$

38. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges?

(A) $-1 \le x \le 1$ (B) $-1 < x \le 1$ (C) $-1 \le x < 1$

(D) $-1 < x < 1$ (E) All real x

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39. If $\frac{dy}{dx} = y \sec^2 x$ and $y = 5$ when $x = 0$, then $y =$

(A) $e^{\tan x} + 4$ (B) $e^{\tan x} + 5$ (C) $5e^{\tan x}$

(D) $\tan x + 5$ (E) $\tan x + 5e^x$

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40. Let f and g be functions that are differentiable everywhere. If g is the inverse function of f and if $g(-2) = 5$ and $f'(5) = -\frac{1}{2}$, then $g'(-2) =$

(A) 2 (B) $\frac{1}{2}$ (C) $\frac{1}{5}$ (D) $-\frac{1}{5}$ (E) -2

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41. $\lim_{n\to\infty} \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \ldots + \sqrt{\frac{n}{n}}\right] =$

(A) $\frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (B) $\int_0^1 \sqrt{x}\,dx$ (C) $\int_0^1 x\,dx$

(D) $\int_1^2 x\,dx$ (E) $2\int_1^2 x\sqrt{x}\,dx$

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42. If $\int_1^4 f(x)\,dx = 6$, what is the value of $\int_1^4 f(5-x)\,dx$?

(A) 6 (B) 3 (C) 0 (D) -1 (E) -6

43. Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

(A) $\dfrac{3\ln 3}{\ln 2}$ (B) $\dfrac{2\ln 3}{\ln 2}$ (C) $\dfrac{\ln 3}{\ln 2}$ (D) $\ln\left(\dfrac{27}{2}\right)$ (E) $\ln\left(\dfrac{9}{2}\right)$

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44. Which of the following series converge?

I. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{2n+1}$

II. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{3}{2}\right)^n$

III. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) I, II, and III

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45. What is the area of the largest rectangle that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$?

(A) $6\sqrt{2}$ (B) 12 (C) 24 (D) $24\sqrt{2}$ (E) 36